Motions in a Bose condensate. III. The structure and effective masses of charged and uncharged impurities

1974 J. Phys. A: Math. Nucl. Gen. 7 260

(http://iopscience.iop.org/0301-0015/7/2/012)

J. Phys. A: Math., Nucl. Gen., Vol. 7, No. 2, 1974. Printed in Great Britain. © 1974

Motions in a Bose condensate
III. The structure and effective masses of charged and uncharged impurities

J Grant† and P H Roberts

School of Mathematics, University of Newcastle upon Tyne, Newcastle upon Tyne NE1 7RU, UK

Received 12 March 1973, in final form 24 August 1973

Abstract. On the assumption that the radius b of an uncharged hard impurity (^{3}He) in an imperfect Bose gas is large compared with the healing length a, it is shown that its effective (hydrodynamic) radius $b_{\rm eff}$ is $b+a\surd 2$. It is also shown that, correct to the first *two* orders in the expansion in powers of a/b, its effective (hydrodynamic) mass is $\frac{2}{3}\pi\rho b_{\rm eff}^3$, where ρ is the fluid density. The modifications to these results required when the impurity is charged (as for ^{4}He$_2^+$) are derived. The results are exact within the framework of the theory, and are obtained analytically. Comparisons with the numerical integrations of Padmore and Fetter are made.

The structure of the electron bubble is examined with a theory which treats $\epsilon = (a\mu/lM)^{1/5}$ as small (and μ/M as negligible), where μ is the electron mass, M the boson mass, and l is the electron–boson scattering length. To leading order, the radius of the bubble is shown to be $b = (\pi M^2a^2/\mu\rho)^{1/5}$, and the electron energy is found to be $h^2/8\mu b^2$. The corrections required at the next order in the a/b expansion are given, and it is shown that, even when polarization effects are negligible, $b_{\rm eff}$ is less than b. The effective mass is still, however, closely $\frac{2}{3}\pi\rho b_{\rm eff}^3$, although motion tends to expand the bubble and make it oblate; the ellipticity is evaluated. When polarization effects are included, the radius of the bubble is reduced by about 0·3 healing lengths. Comparisons are made with the numerical calculations of Clark.

1. Introduction

It has become increasingly apparent over the past decade that the deliberately introduced impurity can be a fruitful experimental probe of the structure and behaviour of helium II. These impurities are of three types: neutral atoms such as ^{3}He, positive ions such as ^{4}He$_2^+$, and negative ions such as electrons. The first two are 'hard' impurities of radii about 4 Å and 8 Å respectively; the light electron, through the energy of its motion, carves out a 'soft' bubble of about 16 Å radius from the surrounding helium. In all cases, the induced hydrodynamic mass of the impurity is comparable with, or greater than, its physical mass. It is one of the objectives of the theory to compute this induced mass. Also, the flow at great distances from a moving ion resembles that of a sphere of a radius $b_{\rm eff}$, different from its actual physical radius b. It is another objective of the theory to calculate this effective radius. The current experimental and theoretical situation has been recently comprehensively reviewed by Fetter (1974).

† Present address: Aerodynamics Division, RAE, Farnborough, UK.

In the previous two papers of this series (Roberts and Grant 1971, Grant 1971), the authors have used the Bose condensate model in an attempt to throw added light on the structures and oscillation spectra of vortex lines and rings in helium II. The same model is used here to study the structure of the ions. In this sense, the paper continues work by Gross (1966), Clark (1965, 1966), and Padmore and Fetter (1971, to be referred to as PF).

Gross (1966) proposed two models for an impurity in a condensate. The first, which he termed 'the instantaneous wavefunction' approximation, is the more suitable for the (comparatively) massive impurity, such as ^{3}He. It supposes that the probability of finding a boson at a given position depends on the precise value of the impurity co-ordinates at that moment. The second, which he termed 'the self-trapping wavefunction' approximation, is the more suitable for the light impurity, such as the electron. It supposes that the impurity wavefunction is determined by the precise positions of the bosons at that instant. PF have used the first of these approximations to evaluate numerically the effective radius and the induced mass of the impurity. We obtain the same quantities analytically, using an expansion technique based on the smallness of $\epsilon = a/b$, the ratio of the healing length a to the impurity radius b. Our method is expounded in the appendix, where comparisons are made with the results of PF. We find that our values, which are calculated from elementary algebraic expressions, are not more than a few per cent from theirs, in the cases of prime physical interest.

Our results for the hard impurity are, in fact, readily summarized. The uncharged impurity has an effective hydrodynamic radius, b_{eff}, of $b+a\sqrt{2}$, and an induced mass, m_{ind}, greater by a factor of $\mathscr{M} = (b+3a\sqrt{2})/b$ than the value, $m_{\mathrm{cl}} = \frac{2}{3}\pi\rho b^3$, predicted by classical fluid mechanics, where ρ is the mass density. In the case of the positive ion, we introduce the dimensionless parameter

$$\alpha = \frac{\tilde{\alpha} m^* Z^2 e^2 a}{\hbar^2 b^3}, \tag{1.1}$$

where $\tilde{\alpha}$ is the atomic polarizability, Ze is the ionic charge (in esu), and m^* is the ionic mass, modified to allow for recoil effects. We find that b_{eff} is reduced to $b+a\sqrt{2}-5a\alpha/7$, and that $\mathscr{M} = 1+(3\sqrt{2}-10\alpha/7)(a/b)$.

In our opinion, the simplicity of such easily computed expressions for b_{eff} and m_{ind}, and indeed the elementary nature of the method by which they were derived, constitute an advantage over the numerical and variational approaches (provided, of course, that ϵ is small). Our expansions in ϵ are systematic and exact, the errors involved in truncation at any level being readily estimated, in contrast to variational methods from which eigenfunctions are often poorly given and for which it is often difficult to estimate how far the bound obtained for the eigenvalue differs from its true value. Also, in contrast to the numerical approach, results are not obtained on a 'case to case' basis, and an appreciation of the physical structure of the solution is forthcoming.

The self-trapping theory is considerably more complex. Gross (1966) identified the three principal contributions to the energy,

$$\mathscr{E}_E = \frac{\hbar^2\pi^2}{2\mu b^2} + \frac{2\pi V_0 \rho^2 b^3}{3M^2} + 4\pi T b^2, \tag{1.2}$$

of the stationary electron. The first is the ground state energy E_e of a particle of mass μ in a spherical container having infinite potential walls; the second is the energy required to dig such a cavity in the condensate in the face of the short-range repulsive potential V_0; the third arises from the tension T of the interface between electron and condensate.

By studying a simple plane model, Gross was able to estimate that, on the condensate picture,

$$T \simeq \frac{\sqrt{2}\hbar^2\rho}{3M^2a}. \tag{1.3}$$

In (1.2) and (1.3), M is the helium mass.

The ratio of the surface tension energy to the cavity energy is, by (1.3), $4a\sqrt{2}/b$. Thus when μ is large (b small), the surface tension energy dominates the cavity energy. On minimizing $\mathscr{E}_E$ with respect to b, Gross found that $\mathscr{E}_E \simeq \hbar^2\pi^2/\mu b^2$, and

$$b^4 \simeq \frac{\hbar^2\pi^2}{8\pi\mu T} = \frac{3M^2a}{8\pi\rho\mu\sqrt{2}}. \tag{1.4}$$

If on the other hand (as in our case) μ is small, the final term in (1.2) is small compared with the term which preceeds it. Then $\mathscr{E}_E \simeq 5\hbar^2\pi^2/3\mu b^2$, and

$$b^5 \simeq \frac{\pi\hbar^2M^2}{2\mu V_0\rho^2} = \frac{\pi M^2a^2}{\mu\rho}. \tag{1.5}$$

Introducing the surface tension as a correction, (1.5) is modified to

$$b = \left(\frac{\pi M^2a^2}{\mu\rho}\right)^{1/5}\left[1-\frac{8\sqrt{2}}{15}\left(\frac{\mu\rho a^3}{\pi M^2}\right)^{1/5}\right]. \tag{1.6}$$

In § 2, we adopt the same technique (expansion for small ϵ) to examine the self-trapping approximation as we used for the instantaneous wavefunction approach. Since ϵ is rather smaller for the negative ion than for other impurities, we have grounds for believing that our results will be even more reliable for the electron bubble than for the hard ions for which, as we have already stated, good agreement with the results of PF has been obtained. In the analysis of § 2, we do indeed confirm the general form of (1.6); in fact, we find that (1.5) is exact to leading order as $\epsilon = a/b \to 0$. A detailed discussion of the healing layer on the surface of the bubble, in which the electron and condensate wavefunctions overlap, shows that the surface tension should not be independent of the electron: the constant $8\sqrt{2}/15$ in (1.6) should be replaced by a function of $q^2 = \mu U_0/MV_0$, where U_0 is the repulsive short-range potential between electron and boson. This function, which can be evaluated only by simple numerical integrations, is given in a few cases. The results are compared with the computations of Clark (1965, 1966), and the agreement is found to be good. Various values of a and of the electron–boson scattering length $l = \mu U_0/2\pi\hbar^2$ are considered, and values of b in the range 15·75 Å to 18·51 Å are obtained (see table 1). It is shown (§ 3) that, when polarization effects are allowed for, these values are reduced by about 0·3 healing lengths.

When the bubble moves with a velocity u, slow compared with the speed of sound, it becomes a slightly oblate spheroid of approximate ellipticity $39(ua/\kappa)^2$, where $\kappa(=h/M)$ is the unit of circulation. It is now found that the effective hydrodynamical radius b_{eff} of the bubble is less than b, but that the classical expression $\frac{2}{3}\pi\rho b^3$ still provides a good estimate of the induced mass provided that b is replaced by b_{eff}. Results for the electron bubble are summarized in table 1 in which NDU stands for non-dimensional units (based on a as length, and ρa^3 as mass), 'm_{ind}' is induced mass, and

$$\mathscr{M} = \frac{m_{\text{ind}}}{\frac{2}{3}\pi\rho b^3}, \qquad \mathscr{M}_{\text{eff}} = \frac{m_{\text{ind}}}{\frac{2}{3}\pi\rho b_{\text{eff}}^3}. \tag{1.7}$$

Table 1. Application of the theory of §§ 2 and 3
(*a*) Electron–boson scattering length $l = 0{\cdot}60$ Å

a(Å)	q	d(Å)	ϵ	b (NDU)	b (Å)	b_{eff} (NDU)	b_{eff} (Å)	m_{ind}	$\frac{2}{3}\pi b^3$	$\frac{2}{3}\pi b^3_{eff}$	$\mathcal{M}$	$\mathcal{M}_{eff}$	E_e (eV)	$\mathcal{E}$ (eV)	Δb (NDU)	Δb (Å)
0·82	0·33	2·71	0·180	19·21	15·75	17·00	13·94	10970	14847	10285	0·74	1·067	0·150	0·284	0·316	0·259
0·90	0·37	2·25	0·183	18·11	16·30	16·13	14·52	9366	12438	8793	0·753	1·065	0·140	0·264	0·318	0·287
1·00	0·41	1·82	0·187	16·93	16·93	15·20	15·20	7820	10162	7357	0·77	1·063	0·130	0·242	0·313	0·313
1·13	0·46	1·43	0·192	15·64	17·67	14·17	16·01	6305	8011	5955	0·79	1·059	0·120	0·220	0·306	0·346
1·20	0·49	1·27	0·194	15·08	18·09	13·74	16·48	5739	7179	5430	0·80	1·057	0·114	0·210	0·303	0·364
1·28	0·52	1·11	0·196	14·46	18·51	13·23	16·93	5118	6335	4849	0·81	1·055	0·109	0·199	0·296	0·379

(*b*) Healing length $a = 1{\cdot}0$ Å

l(Å)	q	d(Å)	ϵ	b (NDU)	b (Å)	b_{eff} (NDU)	b_{eff} (Å)	m_{ind}	$\frac{2}{3}\pi b^3$	$\frac{2}{3}\pi b^3_{eff}$	$\mathcal{M}$	$\mathcal{M}_{eff}$	E_e (eV)	$\mathcal{E}$ (eV)	Δb (NDU)	Δb (Å)
0·45	0·35	1·82	0·198	17·12	17·12	15·04	15·04	7712	10490	7121	0·735	1·083	0·127	0·242	0·305	0·305
0·50	0·37	1·82	0·194	17·05	17·05	15·11	15·11	7764	10380	7223	0·75	1·075	0·128	0·242	0·307	0·307
0·65	0·42	1·82	0·184	16·89	16·89	15·24	15·24	7850	10083	7420	0·75	1·058	0·131	0·242	0·307	0·307
0·70	0·44	1·82	0·181	16·83	16·83	15·27	15·27	7852	9986	7456	0·79	1·053	0·132	0·242	0·309	0·309
0·75	0·45	1·82	0·179	16·80	16·80	15·30	15·30	7890	9931	7514	0·794	1·050	0·1325	0·242	0·308	0·308
0·80	0·47	1·82	0·176	16·77	16·77	15·34	15·34	7919	9882	7565	0·80	1·047	0·133	0·242	0·309	0·309

The last two columns give Δb, the amount by which b must be reduced if polarization effects are included. Also shown are the electron energy E_e, and the creation energy $\mathscr{E}_E$.

2. The moving uncharged bubble

The objectives of this section are to compute the effective radius and induced mass of an uncharged impurity, using the self-trapping wavefunction approximation (Gross 1966). The effect of charge is considered in § 3.

In the Hartree approximation, the problem reduces to that of solving the coupled equations

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\psi+(U_0|\phi|^2+V_0|\psi|^2-E)\psi, \tag{2.1}$$

$$i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2\mu}\nabla^2\phi+(U_0|\psi|^2-E_e)\phi, \tag{2.2}$$

(Gross 1966, Clark 1965, 1966). Here M, E, and ψ are the mass, single particle energy and wavefunction for the bosons; like quantities for the impurity are denoted by μ, E_e and ϕ. The wavefunctions are required to obey the normalized conditions

$$N = \int_\tau |\psi|^2\,\mathrm{d}\tau, \qquad 1 = \int_\tau |\phi|^2\,\mathrm{d}\tau, \tag{2.3}$$

where N is the total number of bosons in the system, which fills volume τ. In what follows, we allow N and τ to become infinite together. In writing (2.1) and (2.2), δ-function forms $U_0\delta(\boldsymbol{x}-\boldsymbol{x}')$ and $V_0\delta(\boldsymbol{x}-\boldsymbol{x}')$ have been assumed for the pseudopotentials describing the repulsion between (respectively) boson and impurity, and boson and boson, where $\boldsymbol{x}$ and $\boldsymbol{x}'$ are their positions. To lowest order, perturbation theory predicts such pseudopotentials, with

$$U_0 = \frac{2\pi l\hbar^2}{\mu}, \qquad V_0 = \frac{4\pi d\hbar^2}{M}, \tag{2.4}$$

where l is the boson–impurity scattering length, and d is the boson diameter (eg Huang 1963, chap 13). The expectation value of the total energy of the system is

$$\mathscr{E} = \frac{\hbar^2}{2M}\int_\tau |\nabla\psi|^2\,\mathrm{d}\tau+\frac{V_0}{2}\int_\tau |\psi|^4\,\mathrm{d}\tau+U_0\int_\tau |\psi|^2|\phi|^2\,\mathrm{d}\tau+\frac{\hbar^2}{2\mu}\int_\tau |\nabla\phi|^2\,\mathrm{d}\tau. \tag{2.5}$$

To cast the theory into dimensionless form, we introduce the healing length a, defined by

$$a = \hbar(2\rho_\infty V_0)^{-1/2} = (8\pi d\psi_\infty^2)^{-1/2}, \tag{2.6}$$

where $\rho_\infty = M\psi_\infty^2 = EM/V_0$ is the mean condensate mass density. Unlike our two previous papers in this series, we do not use a as our unit of length, since the radius of the bubble would then turn out to be 'large', in fact of order a/ϵ, where

$$\epsilon = \left(\frac{4\pi a^3\psi_\infty^2 V_0}{U_0}\right)^{1/5} = \left(\frac{a\mu}{lM}\right)^{1/5}. \tag{2.7}$$

(In application, $\epsilon \simeq 0{\cdot}2$; see below.) Instead, we introduce the transformations

$$r \to \frac{ar}{\epsilon}, \qquad t \to \left(\frac{a^2 M}{\hbar \epsilon^2}\right) t, \qquad \psi \to \psi_\infty \psi, \qquad \phi \to \left(\frac{\epsilon^3}{4\pi a^3}\right)\phi. \tag{2.8}$$

Equations (2.1) to (2.3) give

$$2i\epsilon^2 \frac{\partial \psi}{\partial t} = -\epsilon^2 \nabla^2 \psi + (|\psi|^2 + \epsilon^{-2}|\phi|^2 - 1)\psi, \tag{2.9}$$

$$2i\epsilon^2 \delta \frac{\partial \phi}{\partial t} = -\epsilon^2 \nabla^2 \phi + (q^2|\psi|^2 - \epsilon^2 k_M^2)\phi, \tag{2.10}$$

solutions of which must satisfy

$$\psi \to 1, \qquad \text{for } r \to \infty, \tag{2.11}$$

$$\int_\tau |\phi|^2 \, d\tau = 4\pi, \tag{2.12}$$

where

$$q^2 = \frac{\mu U}{M V_0} = \frac{l}{2d}, \qquad \delta = \frac{\mu}{M}, \qquad \epsilon k_M = \left(\frac{\mu E_e}{M E}\right)^{1/2}. \tag{2.13}$$

In writing (2.11), we have assumed that we are in a frame in which the condensate is at rest at infinity. The constant k_M in (2.13) is a dimensionless measure of the single-particle impurity energy E_e. According to (2.5), the energy of the whole system, measured from the state in which the condensate is uniformly spread over all space, is (in units of $E^2 a^3 / V_0 \epsilon^3$)

$$\mathscr{E}_E = \int_\tau [|\nabla\psi|^2 + \tfrac{1}{2}(1 - |\psi|^2)^2 + |\psi|^2|\phi|^2 + q^{-2}|\nabla\phi|^2] \, d\tau. \tag{2.14}$$

We will suppose that the only disturbance present in the condensate is that caused by the uniform motion of the impurity with velocity u in the positive z direction; we define $U = aMu/\hbar\epsilon$ to be its dimensionless speed. Though ψ and ϕ are not time independent, z and t must always appear in the combination $z - Ut$, since the disturbance preserves its form as it travels in the z direction. Thus, we may replace $\partial/\partial t$ by $-U\partial/\partial z$ in (2.9) and (2.10), to obtain

$$\epsilon^2 \nabla^2 \psi = (|\psi|^2 + \epsilon^{-2}|\phi|^2 - 1)\psi + 2i\epsilon^2 U \frac{\partial \psi}{\partial z}, \tag{2.15}$$

$$\epsilon^2 \nabla^2 \phi = (q^2|\psi|^2 - \epsilon^2 k_M^2)\phi + 2i\epsilon^2 \delta U \frac{\partial \phi}{\partial z}. \tag{2.16}$$

Equation (2.15) may be reduced to fluid mechanical form by writing

$$\psi = R \exp(iS), \tag{2.17}$$

where R and S are real (eg Roberts and Grant 1971, § 2). Then R^2 may be regarded as the fluid density, and S as the velocity potential of the flow. Because of the small size of $\delta(\simeq 1{\cdot}4 \times 10^{-4})$ for the application to the electron in helium which we have in mind, we neglect the last term in (2.16), and can therefore, without loss of generality, assume that ϕ is real. On dividing (2.15) into its real and imaginary parts, we obtain

$$\epsilon^2[\nabla^2 R - R(\nabla S)^2] = (R^2 + \epsilon^{-2}\phi^2 - 1)R - 2\epsilon^2 U R\frac{\partial S}{\partial z}, \tag{2.18}$$

$$R\nabla^2 S + 2\nabla R\,.\,\nabla S = 2U\frac{\partial R}{\partial z}, \tag{2.19}$$

$$\epsilon^2\nabla^2\phi = (q^2R^2 - \epsilon^2 k_M^2)\phi. \tag{2.20}$$

We will seek solutions valid for small U. Without loss of generality, we assume

$$R = R_0(r) + U^2[R_{20}(r) + R_{22}(r)P_2(\cos\theta)] + \ldots, \tag{2.21}$$

$$S = US_1(r)P_1(\cos\theta) + \ldots, \tag{2.22}$$

$$\phi = \phi_0(r) + U^2[\phi_{20}(r) + \phi_{22}(r)P_2(\cos\theta)] + \ldots, \tag{2.23}$$

where (r, θ, χ) are spherical coordinates, with origin at the centre of the bubble and $\theta = 0$ along $\boldsymbol{u}$. Substituting (2.21) to (2.23) in (2.18) to (2.20) and equating like powers of U, we obtain a sequence of coupled ordinary differential equations, of which the first few are

$$\epsilon^2\left(\frac{d^2R_0}{dr^2} + \frac{2}{r}\frac{dR_0}{dr}\right) = (R_0^2 + \epsilon^{-2}\phi_0^2 - 1)R_0, \tag{2.24}$$

$$\epsilon^2\left(\frac{d^2\phi_0}{dr^2} + \frac{2}{r}\frac{d\phi_0}{dr}\right) = (q^2R_0^2 - \epsilon^2 k_M^2)\phi_0, \tag{2.25}$$

$$R_0\left(\frac{d^2S_1}{dr^2} + \frac{2}{r}\frac{dS_1}{dr} - \frac{2S_1}{r^2}\right) + 2\frac{dR_0}{dr}\frac{dS_1}{dr} = 2\frac{dR_0}{dr}, \tag{2.26}$$

$$\epsilon^2\left\{\left(\frac{d^2R_{20}}{dr^2} + \frac{2}{r}\frac{dR_{20}}{dr}\right) - \tfrac{1}{3}R_0\left[\left(\frac{dS_1}{dr}\right)^2 + 2\left(\frac{S_1}{r}\right)^2\right]\right\}$$
$$= (3R_0^2 + \epsilon^{-2}\phi_0^2 - 1)R_{20} + 2\epsilon^{-2}R_0\phi_0\phi_{20} - \tfrac{2}{3}\epsilon^2 R_0\left(\frac{dS_1}{dr} + \frac{2S_1}{r}\right), \tag{2.27}$$

$$\epsilon^2\left[\frac{d^2\phi_{20}}{dr^2} + \frac{2}{r}\frac{d\phi_{20}}{dr}\right] = q^2(2R_0\phi_0R_{20} + R_0^2\phi_{20}) - \epsilon^2 k_M^2\phi_{20}. \tag{2.28}$$

On substituting (2.21) to (2.23) into (2.14), performing the angular integrations, carrying out some integrations by parts, and neglecting terms of order U^4 and smaller, we obtain

$$\mathscr{E}_E = 4\pi\int_0^\infty \left[q^{-2}\left(\frac{d\phi_0}{dr}\right)^2 + \tfrac{1}{2}(1 - R_0^4)\right]r^2\,dr$$
$$+ \tfrac{8}{3}\pi U^2\int_0^\infty \left(3k_M^2 q^{-2}\phi_0\phi_{20} - S_1R_0\frac{dR_0}{dr}\right)r^2\,dr. \tag{2.29}$$

In considering how (2.24) to (2.28) may be solved, we recall that the principal application we have in mind is the electron bubble in helium. If we take $\rho_\infty = 0{\cdot}145\,\mathrm{g\,cm^{-3}}$ and $d = 2{\cdot}7$ Å (Clark 1966), we obtain $a = 0{\cdot}82$ Å. The work of Burdick (1965) and O'Malley (1963) indicates that $l \simeq 0{\cdot}60$ Å, from which $\epsilon \simeq 0{\cdot}18$. If, alternatively, we take $a = 1{\cdot}28$ Å, as is implied by the work of Rayfield and Reif (1964), we obtain

$d \simeq 1{\cdot}11$ Å and, on again taking $l = 0{\cdot}60$ Å, we obtain $\epsilon \simeq 0{\cdot}20$. Both of these estimates suggest that good results will be obtained from a theory which treats ϵ as small. Such a theory is developed below. It is supposed throughout that $q = O(1)$ in the limit $\epsilon \to 0$. Using the data just quoted, we find that $q \simeq 0{\cdot}33$ for $d = 2{\cdot}7$ Å, and $q \simeq 0{\cdot}52$ for $d = 1{\cdot}11$ Å.

The term $\epsilon^{-2}|\phi|^2\psi$ on the right-hand side of (2.15) apparently dominates in the limit $\epsilon \to 0$. We may distinguish three possibilities defining three distinct regimes of the solution: (i) a 'mainstream condensate', (ii) a 'mainstream impurity', and (iii) a 'boundary layer', between them. In (i), the left-hand side of (2.15) is negligible, but (despite appearances) $\epsilon^{-2}|\phi|^2\psi$ is not large: the impurity wavefunction ϕ is too small. Similarly in (ii), the condensate wavefunction ψ vanishes to leading order, and $\epsilon^{-2}|\phi|^2\psi$ does not dominate other terms in (2.15). In (iii), it transpires that $\phi = O(\epsilon)$ and $d/dr = O(1/\epsilon)$. Thus, $\epsilon^{-2}|\phi|^2\psi$ is of the same order as other terms, on the right-hand side of (2.15), and also as its left-hand side. The problem reduces to that of constructing acceptable solutions in (i) and (ii), and matching them through a solution of (iii). Mathematically, the problem is one of singular perturbation theory, and is of a type frequently encountered in the study of high Reynolds number flows in fluid mechanics, from which the terms 'mainstream' and 'boundary layer' used above were drawn. In view of a referee's comment on the first version of this paper, we wish to emphasize that the matching procedure is not merely one of ensuring the continuity of the solution without considering the continuity or otherwise of its derivatives; it is one of constructing uniformly valid solutions, which together with all their derivatives, are continuous everywhere to the order warranted. The method is expounded in well known texts by Cole (1968) and van Dyke (1964); a conceptually similar procedure was adopted by Langer (1937) in his theory of penetration of potential barriers.

(i) *The mainstream condensate.* Assuming that $d/dr = O(1)$, the dominant terms of (2.24) and (2.27) vanish if $\phi = 0$, ie if

$$\phi_0 = \phi_{20} = 0. \tag{2.30}$$

The remaining O(1) terms of these equations, together with (2.26), give

$$R_0 = 1, \qquad R_{20} = -\epsilon^2 C^2 r^{-6}, \qquad S_1 = -Cr^{-2}, \tag{2.31}$$

where C is a constant.

To leading order, (2.22) and the last of (2.31) give

$$S = -CUr^{-2}\cos\theta, \tag{2.32}$$

which should be compared with the classical (dimensionless) result,

$$S = -\tfrac{1}{2}Ub_{\text{eff}}^3 r^{-2}\cos\theta, \tag{2.33}$$

for the potential flow created by a moving sphere of radius b_{eff}. This suggests that we might profitably introduce the effective hydrodynamic radius b_{eff} of the ion by writing (see PF)

$$b_{\text{eff}}^3 = 2C. \tag{2.34}$$

(ii) *The mainstream impurity.* Assuming again that $d/dr = O(1)$, we now make the dominant terms of (2.24) and (2.27) vanish by setting $\psi = 0$, ie by taking

$$R_0 = R_{20} = 0. \tag{2.35}$$

The equations governing ϕ_0, ϕ_{20} and (in fact) ϕ_{22} reduce in leading order to Helmholtz equations, to which the physically acceptable solutions are

$$\phi_0 = Aj_0(k_M r), \qquad \phi_{20} = Bj_0(k_M r), \qquad \phi_{22} = Dj_2(k_M r), \tag{2.36}$$

where A, B, and D are constants, $j_\nu(z)$ is the spherical Bessel function $(\pi/2z)^{1/2}J_{\nu+\frac{1}{2}}(z)$, and $J_\nu(z)$ is the Bessel function of the first kind, of order ν and argument z.

We may define the edge of the impurity bubble to be the locus, $r(\theta, \chi)$, of the smallest values of r on which ϕ vanishes. By (2.23) and (2.36), we see that its equation is that of the spheroid

$$r = b\left[1+U^2\left(\frac{D}{A}\right)j_2(\pi)P_2(\cos\theta)\right], \tag{2.37}$$

where $b = \pi/k_M$. It can be shown, by an analysis similar to that given below, that D/A is negative. Since $j_2(\pi) > 0$, it follows that motion makes the bubble oblate. In fact, its ellipticity is approximately $0{\cdot}98\,\epsilon^2U^2$. We also observe that the quantum radius b is not necessarily the same as the effective radius b_{eff} introduced earlier. We will find, however, that the two quantities are equal, to leading order in ϵ.

In describing the surface (2.37) as 'the edge of the bubble', we do not wish to imply that ϕ is nonzero only for points within (2.37), or even that (2.36) applies with uniform validity up to the surface (2.37). The boundary layer that adjusts the mainstream condensate and impurity solutions in fact contains (2.37). If, ignoring this complication, we suppose that ϕ is nonzero only within (2.37), we find that (2.23), (2.36) and the normalization condition (2.12) give

$$A^2+2ABU^2 = \frac{2k_M^3}{\pi}. \tag{2.38}$$

Because of the boundary layer, this result is not precise. Since, however, the error involved is only of order ϵ^3, which is higher than the order to which we will in fact work, we will accept (2.38). Thus, while the condensate is not actually an abrupt potential barrier to the impurity, we may ignore this fact when applying the normalization condition to order ϵ^2.

(iii) *The boundary layer.* Our earlier discussion suggests that between regimes (i) and (ii) there exists a boundary, or 'healing', layer which matches the solutions in the two mainstreams together. It suggests that we should introduce the stretched coordinate ξ defined by

$$r = b+\epsilon\xi = \frac{\pi}{k_M}+\epsilon\xi, \tag{2.39}$$

and write

$$R_0 = R(\xi), \qquad R_{20} = H(\xi), \qquad S_1 = S(\xi), \qquad \phi_0 = \epsilon X(\xi), \qquad \phi_2 = \epsilon T(\xi). \tag{2.40}$$

In examining the boundary layer, it is convenient to expand the functions (2.40), and constants such as A, B and C, in power series in ϵ. We will not expand k_M in such a way, but will suspend the normalization condition (2.38) to the end of the calculation. On finally substituting our solution into (2.38), we will then recognize that $k_M = k_M(\epsilon, U^2)$. We will write

$$R = R_0+\epsilon R_1+\epsilon^2R_2+\dots, \tag{2.41}$$

and expand H, S, X, T, A, B and C similarly. (The new significance of the suffixes 0, and 1 in R_0 and S_1 should not cause confusion.) Substituting series of the form (2.41) into (2.24) to (2.28) and equating like powers of ϵ, we obtain a sequence of equations amongst which are

$$\frac{d^2R_0}{d\xi^2} = (R_0^2 + X_0^2 - 1)R_0, \tag{2.42}$$

$$\frac{d^2X_0}{d\xi^2} = q^2R_0^2X_0. \tag{2.43}$$

These must be solved numerically but, once this has been done, the solution to order ϵ^2 can, in fact, be completed in terms of R_0 and X_0.

Correct to order ϵ, (2.24) to (2.26) give

$$\frac{d^2R}{d\xi^2} + \frac{2\epsilon}{b}\frac{dR}{d\xi} = (R^2 + X^2 - 1)R, \tag{2.44}$$

$$\frac{d^2X}{d\xi^2} + \frac{2\epsilon}{b}\frac{dX}{d\xi} = q^2R^2X, \tag{2.45}$$

$$R\left(\frac{d^2S}{d\xi^2} + \frac{2\epsilon}{b}\frac{dS}{d\xi}\right) + 2\frac{dR}{d\xi}\frac{dS}{d\xi} = 2\epsilon\frac{dR}{d\xi}. \tag{2.46}$$

Multiplying (2.44) by $dR/d\xi$, (2.45) by $q^{-2}\,dX/d\xi$ and adding, we see that for any ξ_1 and ξ_2,

$$\left[\left(\frac{dR}{d\xi}\right)^2 + q^{-2}\left(\frac{dX}{d\xi}\right)^2 - \tfrac{1}{2}R^4 + R^2 - R^2X^2\right]_{\xi_2}^{\xi_1} = -\frac{4\epsilon}{b}\int_{\xi_2}^{\xi_1}\left[\left(\frac{dR}{d\xi}\right)^2 + \frac{1}{q^2}\left(\frac{dX}{d\xi}\right)^2\right]d\xi. \tag{2.47}$$

By (2.31) and (2.36) we have

$$R(\xi_1) \to 1, \qquad X(\xi_1) \to 0, \qquad R(\xi_2) \to 0, \qquad X(\xi_2) \to -\frac{A\xi_2}{b} + A\epsilon\left(\frac{\xi_2}{b}\right)^2, \tag{2.48}$$

as $\xi_1 \to +\infty$ and $\xi_2 \to -\infty$. Thus, (2.47) gives, correct to order ϵ,

$$A = \frac{qb}{\sqrt{2}}\left\{1 + \frac{4\epsilon}{b}⨍_{-\infty}^{\infty}\left[\left(\frac{dR_0}{d\xi}\right)^2 + \frac{1}{q^2}\left(\frac{dX_0}{d\xi}\right)^2\right]d\xi\right\}, \tag{2.49}$$

where the bar through the integral sign signifies that the finite part of the integral is taken.

To leading order, (2.46) shows that $R^2\,dS/d\xi$ is constant, but the condition of zero mass flux into the bubble ($\xi \to -\infty$) implies that that constant is zero. To order ϵ, (2.46) therefore gives

$$R^2\frac{dS}{d\xi} = \epsilon R^2 + \beta, \tag{2.50}$$

where β is a constant which, again by the condition of zero mass flux, is zero. It follows that, to order ϵ,

$$S = \epsilon\xi + \gamma, \tag{2.51}$$

where γ is a constant. Matching this solution to (2.31), we obtain, to order ϵ,

$$C = \tfrac{1}{2}b^3, \qquad \gamma = -\frac{C}{b^2} = -\tfrac{1}{2}b, \tag{2.52}$$

the first of which shows that, to this order, b and $b_{\rm eff}$ coincide (cf equation (2.34)).

It is interesting to note that the form (2.51) of the condensate potential yields velocity components that diverge inside the bubble as $\xi \to -\infty$. There is, however, no cause for alarm; the condensate density is exponentially small inside the bubble and thus there is no mass flux for large $|\xi_2|$. The situation is similar to that arising near the axis of a rectilinear vortex in the condensate (eg Roberts and Grant 1971): here the velocity diverges as $\tilde{\omega}^{-1}$ as the distance $\tilde{\omega}$ from the axis approaches zero, but the density decreases as $\tilde{\omega}^2$.

Making use of (2.51) and (2.52), we find that (2.26) gives

$$R_0\frac{d^2S_2}{d\xi^2}+2\frac{dR_0}{d\xi}\frac{dS_2}{d\xi} = -\frac{3R_0}{b}, \tag{2.53}$$

showing that

$$R_0^2\frac{dS_2}{d\xi} = -\frac{3}{b}\int_{-\infty}^{\xi} R_0^2(\xi'')\,d\xi''. \tag{2.54}$$

A further integration gives

$$S_2(\xi) = -\frac{3}{b}\int_{-\infty}^{\xi}\frac{d\xi'}{R_0^2(\xi')}\int_{-\infty}^{\xi'} R_0^2(\xi'')\,d\xi''+\gamma', \tag{2.55}$$

where γ' is a constant. Matching this to (2.31), we obtain

$$C_1 = -\frac{3}{2}b^2 ⨍_{-\infty}^{\infty} R_0^2(\xi)\,d\xi, \tag{2.56}$$

which, taken with (2.34), shows that the effective radius is

$$b_{\rm eff} = b-\epsilon ⨍_{-\infty}^{\infty} R_0^2(\xi)\,d\xi. \tag{2.57}$$

On using (2.39) and (2.40) to express (2.27) and (2.28) in boundary layer form we obtain

$$\frac{d^2H}{d\xi^2}+\frac{2\epsilon}{b+\epsilon\xi}\frac{dH}{d\xi}-(3R^2+X^2-1)H-2RXT$$
$$= \frac{1}{3}R\left[\left(\frac{dS}{d\xi}\right)^2-2\epsilon\frac{dS}{d\xi}+2\left(\frac{\epsilon S}{b+\epsilon\xi}\right)^2-4\left(\frac{\epsilon^2 S}{b+\epsilon\xi}\right)\right], \tag{2.58}$$

$$\frac{d^2T}{d\xi^2}+\frac{2\epsilon}{b+\epsilon\xi}\frac{dT}{d\xi}-q^2(R^2T+2RXH)+\epsilon^2k_M^2H = 0. \tag{2.59}$$

By (2.51) and (2.52), the inhomogeneous (source) terms in these equations are of order ϵ^2. It follows that, to order ϵ, only the complementary function can contribute to H and T. Excluding the singular parts of the solution of the homogeneous equations, we find (by comparison of (2.58) and (2.59) with (2.24) and (2.25)) that

$$H = \nu \frac{\mathrm{d}R}{\mathrm{d}\xi}, \qquad T = \nu \frac{\mathrm{d}X}{\mathrm{d}\xi}, \tag{2.60}$$

where ν is a constant. On matching to (2.31) and (2.36), we can show that $\nu = 0$. Thus, H and T must be of order ϵ^2, and, by (2.51), (2.52), (2.58) and (2.59), we have

$$\frac{\mathrm{d}^2 H_2}{\mathrm{d}\xi^2} - (3R_0^2 + X_0^2 - 1)H_2 - 2R_0 X_0 T_2 = \tfrac{1}{2}R_0, \tag{2.61}$$

$$\frac{\mathrm{d}^2 T_2}{\mathrm{d}\xi^2} - q^2(R_0^2 T_2 + 2R_0 X_0 H_2) = 0. \tag{2.62}$$

These should be compared with the equations obtained by differentiating (2.42) and (2.43), namely

$$\frac{\mathrm{d}^2 R_0'}{\mathrm{d}\xi^2} - (3R_0^2 + X_0^2 - 1)R_0' - 2R_0 X_0 X_0' = 0, \tag{2.63}$$

$$\frac{\mathrm{d}^2 X_0'}{\mathrm{d}\xi^2} - q^2(R_0^2 X_0' + 2R_0 X_0 X_0') = 0, \tag{2.64}$$

where $R_0' = \mathrm{d}R_0/\mathrm{d}\xi$ and $X_0' = \mathrm{d}X_0/\mathrm{d}\xi$. This suggests the following manipulation. Multiply (2.61) to (2.64) by respectively $R_0, q^{-2}T_0, -H_2, -q^{-2}T_2$, and add. The result is an exact differential, which can be integrated to give

$$\left[\left(\frac{\mathrm{d}H_2}{\mathrm{d}\xi}\frac{\mathrm{d}R_0}{\mathrm{d}\xi} - H_2\frac{\mathrm{d}^2 R_0}{\mathrm{d}\xi^2}\right) + \frac{1}{q^2}\left(\frac{\mathrm{d}T_2}{\mathrm{d}\xi}\frac{\mathrm{d}X_0}{\mathrm{d}\xi} - T_2\frac{\mathrm{d}^2 X_0}{\mathrm{d}\xi^2}\right) - \frac{1}{4}R_0^2\right]_{\xi_2}^{\xi_1} = 0. \tag{2.65}$$

On taking the limits, $\xi_1 \to \infty$ and $\xi_2 \to -\infty$, we obtain

$$B = \epsilon^2 B_2 = -\frac{qb\epsilon^2}{2\sqrt{2}}. \tag{2.66}$$

This completes the solution as far as we require it. We will note however that the energy (2.29) can be divided into mainstream and boundary layer contributions, the last of which can be expressed in powers of ϵ by using expansions of the form (2.41). On ignoring terms of order ϵ^2 and of order ϵ^2/U^2, we obtain

$$\mathscr{E}_E = \left(\frac{4\pi k_M^2}{q^2} + \frac{2\pi^4}{3k_M^3}\right) + \frac{4\pi^2\epsilon}{k_M^2} \int_{-\infty}^{\infty} \left[\frac{1}{q^2}\left(\frac{\mathrm{d}X_0}{\mathrm{d}\xi}\right)^2 + \frac{1}{2}(1 - R_0^4)\right]\mathrm{d}\xi + \tfrac{2}{3}\pi\epsilon^2 U^2 b_{\mathrm{eff}}^3. \tag{2.67}$$

The coefficient of $U^2/2$ in the expansion of $\mathscr{E}_E$ in powers of U^2 is the induced mass m_{ind} of the impurity (see Gross 1966, equation (2.7)). At first sight, (2.67) might suggest that this is simply $\frac{4}{3}\pi\rho b_{\mathrm{eff}}^3$. It should be noted, however, that the induced mass does not arise solely from the kinetic energy of the fluid (the last term of (2.67)); motion expands the bubble as a whole, ie the k_M appearing in the first two terms of (2.67) depends implicitly on U^2. In fact, from (2.38), (2.49) and (2.66), we obtain

$$k_M = \pi\left(\frac{q}{2\pi}\right)^{2/5}(1 - \tfrac{1}{5}\epsilon^2 U^2) + \frac{8\pi\epsilon}{5}\left(\frac{q}{2\pi}\right)^{4/5} \int_{-\infty}^{\infty} \left[\left(\frac{\mathrm{d}R_0}{\mathrm{d}\xi}\right)^2 + \frac{1}{q^2}\left(\frac{\mathrm{d}X_0}{\mathrm{d}\xi}\right)^2\right]\mathrm{d}\xi, \tag{2.68}$$

and the quantum radius of the bubble is therefore

$$b = \left(\frac{2\pi}{q}\right)^{2/5}(1+\tfrac{1}{5}\epsilon^2U^2) - \frac{8\epsilon}{5}\int_{-\infty}^{\infty}\left[\left(\frac{\mathrm{d}R_0}{\mathrm{d}\xi}\right)^2 + \frac{1}{q^2}\left(\frac{\mathrm{d}X_0}{\mathrm{d}\xi}\right)^2\right]\mathrm{d}\xi, \tag{2.69}$$

which in dimensional units is

$$b = \left(\frac{\pi M^2 a^2}{\mu\rho_\infty}\right)^{1/5}\left[1+\frac{2\pi^2}{5}\left(\frac{au}{\kappa}\right)^2\right] - \frac{8a}{5}\int_{-\infty}^{\infty}\left[\left(\frac{\mathrm{d}R_0}{\mathrm{d}\xi}\right)^2 + \frac{1}{q^2}\left(\frac{\mathrm{d}X_0}{\mathrm{d}\xi}\right)^2\right]\mathrm{d}\xi. \tag{2.70}$$

The presence of the second term on the right-hand side of (2.70) shows that the bubble radius is slightly increased by the passage of condensate over its surface; by Bernoulli's theorem, such a flow creates a reduction in condensate pressure which causes the bubble to expand. The form of (2.70) suggests that, for small ϵ, the present approximation will be a good one provided u is small compared with the speed of sound, $\kappa/2\pi a\sqrt{2}$. It further suggests that, when placed in more general flows (such as that created by a line vortex), it will be admissible to visualize the bubble as a sphere having a radius given by the first and last terms on the right-hand side of (2.70), except when these flows are 'too rapid'. Even a bubble contained within a vortex line may be well represented by the present theory, except possibly for small regions near where line and bubble meet. The method by which Donnelly and Roberts (1969) estimated curvature of the energy well for electron capture by a vortex line would therefore appear to be well founded.

When the expression (2.68) for k_M is substituted into (2.67) and the coefficient of U^2 is extracted, it is found that the dimensionless induced mass of the ion is

$$m_{\mathrm{ind}} = \frac{2\pi}{3}\left(\frac{2\pi}{q}\right)^{6/5} + \frac{2\pi\epsilon}{5}\left(\frac{2\pi}{q}\right)^{4/5}\left[2\int_{-\infty}^{\infty}(1-R_0^4)\,\mathrm{d}\xi - 5\int_{-\infty}^{\infty}R_0^2\,\mathrm{d}\xi \right.$$
$$\left. -16\int_{-\infty}^{\infty}\left(\frac{\mathrm{d}R_0}{\mathrm{d}\xi}\right)^2\mathrm{d}\xi - \frac{12}{q^2}\int_{-\infty}^{\infty}\left(\frac{\mathrm{d}X_0}{\mathrm{d}\xi}\right)^2\mathrm{d}\xi\right]. \tag{2.71}$$

The results of applying this theory to the electron bubble are shown in table 1, where it has been assumed that $\rho_\infty = 0{\cdot}145\,\mathrm{g\,cm^{-3}}$, $\mu = 9{\cdot}109\times10^{-28}\,\mathrm{g}$ and $M = 6{\cdot}648\times10^{-24}\,\mathrm{g}$. Several sets of estimates for a and l have been used. Values obtained for the creation energy $\mathscr{E}_E$ of the stationary bubble ($u = 0$), are consistent with the estimates of Clark (1966), who from a variational calculation obtained the value 0·32 eV as an upper bound. The results of a typical integration of (2.42) and (2.43) are shown in figure 1. It may be noted that the integral appearing in (2.57) arises from the shaded area to the left of $\xi = 0$ minus the shaded area to the right. For the case shown, and indeed for all values of q likely to be relevant to liquid helium, the condensate density is relatively larger for $\xi < 0$ than for $\xi > 0$, and the integral arising in (2.57) is positive and of order unity. Thus, the effective hydrodynamic radius of the bubble is *less*, by some 1 to 2 healing lengths, than its quantum radius. Values of b and b_{eff} are given in table 1.

It must, of course, be wondered how much confidence can be accorded to these numerical results, bearing in mind that they were derived from a theory in which ϵ was assumed small, whereas it appears that ϵ may be as large $\frac{1}{5}$ in our application. For this reason, we show briefly in the appendix the result of applying our method to the instantaneous wavefunction approximation, for which PF have undertaken numerical calculations which do not suppose that ϵ is small. Even for ϵ as large as $\frac{1}{3}$, we find that our results for b_{eff} differ from those of PF by rather less than 10 %, while for $\epsilon = \frac{1}{10}$ they agree to within 2 %.

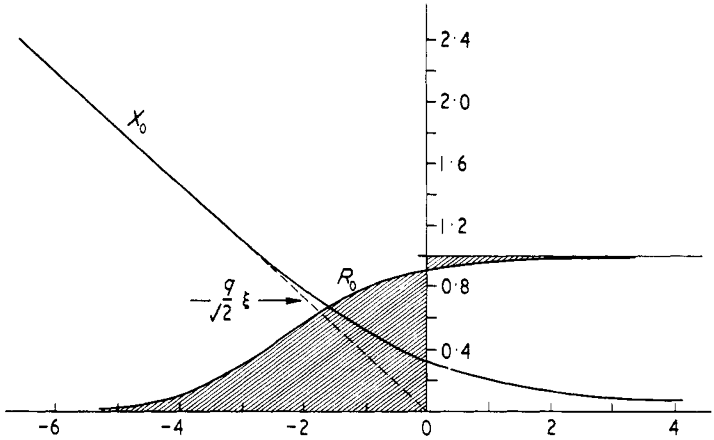

Figure 1. Showing the electron wavefunction X_0 and the condensate wavefunction R_0 obtained by integration of the equations (2.42) and (2.43) for the case $q = 0{\cdot}52$.

3. The stationary charged bubble

The charge on the electron is now recognized. Suppose that a point charge $-e$ is situated at $\boldsymbol{x}_0$ in a medium whose dielectric constant $K(\boldsymbol{x})$ varies slightly with position, ie

$$K(\boldsymbol{x}) = K_0 + f(\boldsymbol{x}), \tag{3.1}$$

where $K_0 = K(\boldsymbol{x}_0)$ and $|f| \ll K_0$. In electrostatic units, the potential, $\Omega(\boldsymbol{x})$, obeys

$$\nabla . (K\nabla\Omega) = 4\pi e\delta(\boldsymbol{x}-\boldsymbol{x}_0). \tag{3.2}$$

Solving (3.2) by iteration, we obtain, to order f/K_0,

$$\Omega(\boldsymbol{x}) = -\frac{e}{K_0|\boldsymbol{x}-\boldsymbol{x}_0|}+\frac{e}{4\pi K_0^2}\int\frac{\partial f(\boldsymbol{x}')}{\partial x_i'}\frac{\partial}{\partial x_i'}\left(\frac{1}{|\boldsymbol{x}'-\boldsymbol{x}_0|}\right)\frac{\mathrm{d}\tau'}{|\boldsymbol{x}'-\boldsymbol{x}|}. \tag{3.3}$$

Since $f(\boldsymbol{x}_0) = 0$, this may be written as

$$\Omega(\boldsymbol{x}) = -\frac{e}{K_0|\boldsymbol{x}-\boldsymbol{x}_0|}+\frac{e}{4\pi K_0^2}\int\frac{(\boldsymbol{x}'-\boldsymbol{x}_0) . (\boldsymbol{x}'-\boldsymbol{x})}{|\boldsymbol{x}'-\boldsymbol{x}_0|^3|\boldsymbol{x}'-\boldsymbol{x}|^3}[K(\boldsymbol{x}')-K(\boldsymbol{x}_0)]\,\mathrm{d}\tau'. \tag{3.4}$$

The integral on the right of (3.4) represents the potential, $\Omega_{\mathrm{p}}(\boldsymbol{x}, \boldsymbol{x}_0)$ created at $\boldsymbol{x}$ by the polarization charges induced by the electron. Its contribution to the potential energy of the electron is

$$V_{\mathrm{p}}(\boldsymbol{x}_0) = -\tfrac{1}{2}e\Omega_{\mathrm{p}}(\boldsymbol{x}_0, \boldsymbol{x}_0), \tag{3.5}$$

or

$$V_{\mathrm{p}}(\boldsymbol{x}) = -\frac{e^2}{8\pi K_0^2}\int\frac{K(\boldsymbol{x}')\,\mathrm{d}\tau'}{|\boldsymbol{x}-\boldsymbol{x}'|^4}. \tag{3.6}$$

Recalling that, in terms of the polarizability $\tilde{\alpha}$ of the helium atom,

$$K(\boldsymbol{x}) = 1+\tilde{\alpha}|\psi(\boldsymbol{x})|^2, \tag{3.7}$$

and that $|\phi(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}$ is the probability that the electron is situated within $\mathrm{d}\boldsymbol{x}$ at $\boldsymbol{x}$, we see that the term

$$-\frac{\tilde{\alpha}e^2}{8\pi}\iint\frac{|\phi(\boldsymbol{x})|^2|\psi(\boldsymbol{x}')|^2}{|\boldsymbol{x}-\boldsymbol{x}'|^4}\,\mathrm{d}\tau\,\mathrm{d}\tau' \tag{3.8}$$

must be added to the energy (2.5) of the system. On performing independent variations with respect to ψ and ϕ, we obtain two new contributions from (3.8) to the equations (2.15) and (2.16), and after scaling in the manner of § 2, we obtain

$$\epsilon^2\nabla^2\psi(\boldsymbol{x}) = \left(|\psi(\boldsymbol{x})|^2+\epsilon^{-2}|\phi(\boldsymbol{x})|^2-1-\frac{\alpha\epsilon}{4\pi}\int\frac{|\phi(\boldsymbol{x}')|^2}{|\boldsymbol{x}-\boldsymbol{x}'|^4}\,\mathrm{d}\tau'\right)\psi(\boldsymbol{x}), \tag{3.9}$$

$$\epsilon^2\nabla^2\phi(\boldsymbol{x}) = \left(q^2|\psi(\boldsymbol{x})|^2-\epsilon^2k^2-\frac{\alpha q^2\epsilon^3}{4\pi}\int\frac{|\psi(\boldsymbol{x}')|^2}{|\boldsymbol{x}-\boldsymbol{x}'|^4}\,\mathrm{d}\tau'\right)\phi(\boldsymbol{x}), \tag{3.10}$$

where the dimensionless polarizability is

$$\alpha = \frac{\tilde{\alpha}Me^2a}{4\pi b^3\hbar^2}. \tag{3.11}$$

The problem of solving (3.9) and (3.10) for the general case of a moving bubble is intricate, and we confine attention to the structure of the stationary electron bubble in the limit of small ϵ. We will assume that α is O(1) as $\epsilon\to 0$. Since spherical symmetry obtains we may take ψ and ϕ to be real, and for consistency of notation we denote ψ by $R(r)$. Equations (3.9) and (3.10) then reduce to

$$\epsilon^2\left(\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}R(r)}{\mathrm{d}r}\right) = \left(R^2(r)+\epsilon^{-2}\phi^2(r)-1-\alpha\epsilon\int_0^\infty\frac{\phi^2(r')r'^2\,\mathrm{d}r'}{(r^2-r'^2)^2}\right)R(r), \tag{3.12}$$

$$\epsilon^2\left(\frac{\mathrm{d}^2\phi(r)}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}\phi(r)}{\mathrm{d}r}\right) = \left(q^2R^2(r)-\epsilon^2k^2-\alpha q^2\epsilon^3\int_0^\infty\frac{R^2(r')r'^2\,\mathrm{d}r'}{(r^2-r'^2)^2}\right)\phi(r). \tag{3.13}$$

To recognize that the electron energy is changed by polarization, we have replaced k_M by k.

The mainstream condensate solution replacing (2.31) can be obtained to order ϵ, by substituting the zero order expression ϕ_0 for ϕ (from the first of (2.36)) into the integral on the right-hand side of (3.12), and setting to zero the bracket in which it lies. We find

$$R(r) = 1+\frac{\alpha\epsilon A^2}{2k^2}\int_0^{\pi/k}\frac{\sin^2kr'}{(r^2-r'^2)^2}\,\mathrm{d}r'. \tag{3.14}$$

The mainstream electron solution is also obtained by iteration about ϕ_0:

$$\phi = \phi_0+\epsilon\phi_1+\ldots. \tag{3.15}$$

On substituting the zero order expression ($R_0 = 1$) in the integral on the right-hand side of (3.13), we obtain

$$\frac{\mathrm{d}^2\phi_1(r)}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}\phi_1(r)}{\mathrm{d}r}+k^2\phi_1(r) = -\alpha q^2\phi_0(r)\int_{\pi/k}^\infty\frac{r'^2\,\mathrm{d}r'}{(r^2-r'^2)^2}, \tag{3.16}$$

from which we find

$$\phi_1(r) = -\frac{\alpha q^2A}{2kr}\int_0^r f(r')\sin k(r-r')\,\mathrm{d}r', \tag{3.17}$$

where

$$f(r) = \frac{\sin kr}{kr}\left[\frac{\pi kr}{\pi^2 - k^2r^2} + \ln\left(\frac{\pi + kr}{\pi - kr}\right)\right]. \tag{3.18}$$

After numerical quadrature, we obtain

$$\phi_1\left(\frac{\pi}{k}\right) = -\frac{1{\cdot}9117\alpha q^2 A}{2\pi k}, \tag{3.19}$$

$$\phi_1'\left(\frac{\pi}{k}\right) = \frac{2{\cdot}3454\alpha q^2 A}{2\pi}, \tag{3.20}$$

results which are required below.

The boundary layer analysis follows the same line as that given in § 2. On expanding R and ϕ as in (2.41), we again obtain (2.42) and (2.43), but (2.47) is modified by the addition of the term,

$$\alpha\epsilon k^2 A_0^2 R_0^2 \int_0^{\pi/k} \frac{\sin^2 kr'}{(\pi^2 - k^2 r'^2)^2}\,\mathrm{d}r' \simeq \left(\frac{1{\cdot}8758kA_0^2}{2\pi^2}\right)\alpha\epsilon R_0^2, \tag{3.21}$$

between the brackets on the left-hand side. After the subsequent matching, it is found that (2.49) is amended to

$$A = \frac{qb}{\sqrt{2}}\left(1 + \epsilon\left\{\frac{1{\cdot}8758\alpha q^2}{4\pi} + \frac{2{\cdot}3454\alpha q^2}{2k} + \frac{4}{b}\int_{-\infty}^{\infty}\left[\left(\frac{\mathrm{d}R_0}{\mathrm{d}\xi}\right)^2 + \frac{1}{q^2}\left(\frac{\mathrm{d}X_0}{\mathrm{d}\xi}\right)^2\right]\mathrm{d}\xi\right\}\right). \tag{3.22}$$

The final step is that of applying the normalization condition (2.38), which here reduces to

$$A^2\left(\frac{\pi}{2k^3} - \frac{\alpha q^2\epsilon}{k^3}\int_0^{\pi/k} f(r)\sin kr(1 + \cos kr)\,\mathrm{d}r + \ldots\right) = 1. \tag{3.23}$$

We find that k is given by (2.68) with the addition of the term,

$$\tfrac{1}{5}\alpha\epsilon q^2\left[2{\cdot}1894 + 0{\cdot}9379\left(\frac{q}{2\pi}\right)^{2/5}\right], \tag{3.24}$$

on the right-hand side. The expression (2.70) for the dimensional radius is amended by the addition of

$$\Delta b = -\tfrac{1}{5}a\alpha\left[1{\cdot}8758q\left(\frac{q}{2\pi}\right)^{3/5} + 27{\cdot}512\left(\frac{q}{2\pi}\right)^{6/5}\right], \tag{3.25}$$

to the right-hand side.

Comparing with (2.70), we see that the term (3.25) represents a radial contraction of the impurity which arises when polarization effects are included. For the electron in helium at zero pressure, the non-dimensional polarizability α, given by (3.11), is approximately

$$\alpha = \frac{201\epsilon^3}{a^2}. \tag{3.26}$$

The reduction Δb in radius b caused by polarization has been evaluated using (3.26), and is given in the final two columns of table 1. As indeed was argued by Gross (1966), the effects of polarization on a structure as large as the electron bubble are not great.

Acknowledgments

We wish to thank our colleague, Professor N C Freeman, for the interest he took in § 2 of this paper, and for giving our asymptotic solution his blessing. We are grateful to Dr R C Clark for a helpful letter, and to an anonymous referee for suggestions which improved and shortened the exposition.

Appendix. The moving charged hard impurity

The objectives of this appendix are to compute the effective radius and induced mass of an impurity, using the instantaneous wavefunction approximation (Gross 1966). The identical problem has also been examined numerically by PF, using a numerical method. Our approach is quite different. As in § 2 above, we develop the first few terms of the expansion of the exact solution in powers of $\epsilon = a/b$. Unlike § 2 above, we are able to obtain the terms we require exactly, without recourse to numerical work. Comparisons with the results of PF not only suggest that their numerical work was accurate, but also that our series converge rapidly, reliable results being obtained even for ϵ as large as $\frac{1}{3}$ (see below).

In the Hartree approximation, the problem reduces to that of solving

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m^*}\nabla^2\psi+(U+V_0|\psi|^2-E)\psi \tag{A.1}$$

(Gross 1966). Here $m^*(=M\mu/(M+\mu))$, E and ψ are the reduced mass, single particle energy and wavefunction for the bosons, and $U(\boldsymbol{x})$ is the interaction potential between impurity and boson. The impurity is now a hard sphere, ie an infinite potential barrier to the condensate, and we therefore have

$$\psi = 0, \qquad \text{at } r = b, \tag{A.2}$$

while the first of the normalization conditions (2.3) holds as before with the understanding that ψ is nonzero only in that part τ' of τ which is not occupied by the impurity. As described in § 3, the charged impurity creates a polarization in the surrounding condensate. The corresponding potential is

$$U(\boldsymbol{x}) = \frac{-\tilde{\alpha}Z^2e^2}{2K^2r^4}, \tag{A.3}$$

(PF, Gross 1966). Polarization contributes to $\mathscr{E}$, and (2.5) is therefore replaced by

$$\mathscr{E} = \frac{\hbar^2}{2m^*}\int_\tau|\nabla\psi|^2\,\mathrm{d}\tau+\frac{V_0}{2}\int_\tau|\psi|^4\,\mathrm{d}\tau-\frac{\tilde{\alpha}Z^2e^2}{2K^2}\int_\tau\frac{|\psi|^2}{r^4}\,\mathrm{d}\tau. \tag{A.4}$$

To cast the theory in dimensionless form, we introduce a healing length a, defined by (2.6) with M replaced by m^*. This substitution is also made in the definition of U and also in (2.8), where $\epsilon = a/b$. Thus (2.7) is abandoned, the surface of the impurity becomes $r = 1$, and the dimensionless impurity radius is $b = 1$. Equation (A.1) gives

$$2i\epsilon^2\frac{\partial\psi}{\partial t} = -\epsilon^2\nabla^2\psi+(|\psi|^2-\alpha\epsilon r^{-4}-1)\psi, \tag{A.5}$$

where

$$\alpha = \frac{\tilde{\alpha} m^* Z^2 e^2 a}{b^3 \hbar^2 K^2}. \tag{A.6}$$

The nature of the solutions of (A.5) depends principally on the form of α selected in the limit $\epsilon \to 0$. We will suppose that $\alpha = O(1)$, a postulate which does not seem physically unrealistic (see PF).

Choosing the reference frame to be the one in which the condensate is at rest at infinity, (2.11) and (2.19) remain applicable, while (2.18) is replaced by

$$\epsilon^2[\nabla^2 R - R(\nabla S)^2] = (R^2 - 1 - \alpha\epsilon r^{-4})R - 2\epsilon^2 U R\frac{\partial S}{\partial z}. \tag{A.7}$$

Adopting expansions (2.21) and (2.22) we obtain (2.26), while (2.24) and (2.27) are replaced by

$$\epsilon^2\left(\frac{d^2R_0}{dr^2} + \frac{2}{r}\frac{dR_0}{dr}\right) = (R_0^2 - 1 - \alpha\epsilon r^{-4})R_0, \tag{A.8}$$

$$\epsilon^2\left\{\left(\frac{d^2R_{20}}{dr^2} + \frac{2}{r}\frac{dR_{20}}{dr}\right) - \frac{1}{3}R_0\left[\left(\frac{dS_1}{dr}\right)^2 + 2\left(\frac{S_1}{r}\right)^2\right]\right\}$$
$$= (3R_0^2 - 1 - \alpha\epsilon r^{-4})R_{20} - \frac{2}{3}\epsilon^2 R_0\left(\frac{dS_1}{dr} + \frac{2S_1}{r}\right). \tag{A.9}$$

The dimensionless form of (A.4) gives, for the excitation energy, $\mathscr{E}_E$,

$$\mathscr{E}_E(U^2) = \mathscr{E}_E(0) - \frac{8\pi U^2}{3}\int_1^\infty S_1 R_0\left(\frac{dR_0}{dr}\right) r^2\, dr. \tag{A.10}$$

The last term of (A.10), which yields the induced mass, m_{ind}, of the impurity, follows by additional integration from the expression (3.28*b*) of PF; it is slightly more convenient in application than their form.

We now develop a theory for small ϵ which parallels that given in § 2, but is simpler in that every step can be performed analytically. There are now only two regions to consider; (i) the mainstream condensate and (ii) the healing layer. The expressions R_0 and S_1 given in (2.31) are replaced by

$$R_0 = 1 + \tfrac{1}{2}\epsilon\alpha r^{-4} + 3\epsilon^2\alpha r^{-6} - (\tfrac{1}{8}\epsilon^2\alpha^2 - 45\epsilon^5\alpha)r^{-8} + \dots, \tag{A.11}$$

$$S_1 = -Cr^{-2} - \epsilon\alpha r^{-3} - 2\epsilon^3\alpha r^{-5} + \tfrac{4}{7}\epsilon\alpha C r^{-6} + (\tfrac{2}{5}\epsilon^2\alpha^2 - 18\epsilon^5\alpha)r^{-7} + \dots. \tag{A.12}$$

The solution in the healing layer can be expanded as in (2.41), and the sequence of equations obtained from (A.8) and (A.9) can be solved exactly, eg

$$R_0 = \tanh\frac{\xi}{\sqrt{2}} + \epsilon\left[-\frac{\sqrt{2}}{3}\left(1 + \cosh^2\frac{\xi}{\sqrt{2}}\right) - \frac{\alpha}{6}\sinh\sqrt{2}\xi + \frac{2\sqrt{2}}{3}\operatorname{sech}^2\frac{\xi}{\sqrt{2}}\right.$$
$$\left. + \left(\frac{1}{3\sqrt{2}} + \frac{\alpha}{6}\right)\left(\frac{3\xi}{\sqrt{2}}\operatorname{sech}^2\frac{\xi}{\sqrt{2}} + 3\tanh\frac{\xi}{\sqrt{2}} + \sinh\sqrt{2}\xi\right)\right] + \dots. \tag{A.13}$$

We again obtain (2.51), though (2.52) is slightly altered both because our dimensionless b is unity and because of polarization:

$$C = \tfrac{1}{2}, \qquad \gamma = -C - \tfrac{6}{7}\alpha\epsilon. \tag{A.14}$$

Working to order ϵ^2, we again obtain (2.54) and (2.55), and matching to (A.12) we find that

$$C_1 = -\frac{3}{2}\int_0^\infty R_0^2(\xi)\,d\xi - \frac{15\alpha}{14}, \tag{A.15}$$

a result which differs from (2.56) only by polarization effects. Similarly, the expression (2.34) for the effective radius differs from (2.57) only slightly:

$$b_{\text{eff}} = 1 - \left(\int_0^\infty R_0^2(\xi)\,d\xi + \frac{5\alpha}{7}\right). \tag{A.16}$$

We see that, in the case of an uncharged impurity ($\alpha = 0$), $b_{\text{eff}} - b$ is (to the order given in (A.16)) precisely a 'displacement thickness', w, defined as the thickness of a layer of fluid (of the same density as that far from the sphere) which, if placed on the surface of the impurity, would have the same mass as the healing layer actually present, ie in dimensionless units

$$\int_1^{1+w} r^2\,dr = \int_1^\infty [1-\psi_0^2(r)]r^2\,dr, \tag{A.17}$$

(PF, equation (3.16)).

Using (A.13), we may reduce (A.16) to

$$b_{\text{eff}} = 1 + (\sqrt{2} - \tfrac{5}{7}\alpha)\epsilon. \tag{A.18}$$

Direct substitution of our solution in the final term of (A.10) gives, for the induced mass,

$$m_{\text{ind}} = \tfrac{2}{3}\pi b_{\text{eff}}^3(1 + \tfrac{5}{7}\alpha\epsilon + O(\epsilon^2)). \tag{A.19}$$

Thus, to order ϵ, the induced mass of an uncharged impurity coincides with the classical value, $\frac{2}{3}\pi b_{\text{eff}}^3$, of a sphere of radius b_{eff}, rather than the actual radius b.

We now compare our results with those obtained by PF. For uncharged spheres of radii 10 and 3 healing lengths (hl), PF obtained b_{eff} of 11·30 and 4·10 hl. These should be compared with our values of 11·41 and 4·41 hl. For a charged sphere with $\alpha = 20/27$, PF obtained $b_{\text{eff}} = 3{\cdot}86$ for a sphere of 3 hl radius, whereas we obtain $b_{\text{eff}} = 3{\cdot}80$. It may be noted that polarization tends to reduce $b_{\text{eff}} - b$. Indeed, PF exhibited negative b_{eff} in their table 2. This conclusion is in agreement with (A.18). If we apply our result to the extreme case in which $\alpha = 400/27$, the effective radius of a sphere of radius 3 hl is $-6{\cdot}1$, according to (A.18); PF obtained $-5{\cdot}53$ hl.

Turning now to the induced mass, we express our results in terms of $\mathscr{M} = m_{\text{ind}}/m_{\text{cl}}$, the ratio of the induced mass to its classical value, $2\pi/3$, based on the actual radius ($b = 1$) of the sphere. According to (A.19),

$$\mathscr{M} = 1 + (3\sqrt{2} - \tfrac{10}{7}\alpha)\epsilon + O(\epsilon^2). \tag{A.20}$$

For uncharged spheres of radii, 10, 3 and 1 hl, we obtain $\mathscr{M} = 1{\cdot}42$, 2·41 and 5·24, which should be compared with the values $\mathscr{M} = 1{\cdot}40$, 2·20, and 4·48 by PF. For a charged sphere of radius 3 hl and $\alpha = 400/27$, we obtain $\mathscr{M} = 2{\cdot}06$; for a radius of 10 hl and $\alpha = 0{\cdot}4$, we have $\mathscr{M} = 1{\cdot}37$; the values given by PF are 2·10 and 1·31 respectively.

Since PF have provided a thorough discussion of the relationship of the present model to the practical problem of finding the effective masses of the neutral ^{3}He atom and the positive ion, we will leave matters here, after one final remark. Although our dimensionless equations are identical to theirs, PF have neglected the recoil of the impurity, and have used M instead of m^* in casting their theory into dimensionless form The resulting difference in length scale does not appear to be negligible in the case of ^{3}He impurities, for which $m^* \simeq \frac{1}{2}M$.

References

Burdick B 1965 *Phys. Rev. Lett.* **14** 11

Clark R C 1965 *Phys. Lett.* **16** 42–3

—— 1966 *Superfluid Helium* ed J F Allen (Amsterdam: North Holland)

Cole J D 1968 *Perturbation Methods in Applied Mathematics* (Waltham, Mass.: Blaisdell)

Donnelly R J and Roberts P H 1969 *Proc. R. Soc.* A **312** 519–51

van Dyke M 1964 *Perturbation Methods in Fluid Mechanics* (New York: Academic Press)

Fetter A L 1974 *The Physics of Liquid and Solid Helium* ed K H Bennemann and J B Ketterson (New York: Wiley)

Grant J 1971 *J. Phys.* A: *Gen. Phys.* **4** 695–716

Gross E P 1966 *Quantum Fluids* ed D F Brewer (Amsterdam: North Holland)

Huang K 1963 *Statistical Mechanics* (New York: Wiley)

Langer R E 1937 *Phys. Rev.* **51** 669–76

O'Malley J F 1963 *Phys. Rev.* **130** 1070

Padmore T C and Fetter A L 1971 *Ann. Phys., NY* **62** 293–319

Rayfield G W and Reif F 1964 *Phys. Rev.* **136** A1194–208

Roberts P H and Grant J 1971 *J. Phys.* A: *Gen. Phys.* **4** 55–72